An algebraic approach to multidimensional behaviors

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Chapter 4

Regular implementability in the space of compactly supported functions

This chapter extends results on regular implementation in [57] and [63] to the case when the signal space is not an injective cogenerator, for instance, the space $\mathfrak{C}_c$ of compactly supported smooth functions on $\mathbb{R}^n$. In this case the bijective correspondence between behaviors and modules fails to hold; also projections and sums of behaviors need not in general be behaviors. A more general version of implementation is introduced and necessary and sufficient conditions are established for the implementation and regular implementation of a given desired behavior.

4.1 Introduction

In this chapter we consider the problem of regular implementation over the space $\mathfrak{C}_c^\infty(\mathbb{R}^n, \mathbb{K})$ (or just $\mathfrak{C}_c$) of compactly supported functions. Earlier work had treated this problem, both for lumped as well as distributed systems, but only when behaviors were considered in injective cogenerators, for example the space $\mathfrak{D}'$ of distributions on $\mathbb{R}^n$ or the space $\mathfrak{C}^\infty$ of smooth functions [6, 58, 63, 72, 79]. The proofs there relied strongly on several facts – that the projection of a behavior is a behavior or that the sum of two behaviors is also one, for instance – facts that are not longer true when the space, such as $\mathfrak{C}_c$, is not injective. Also important was the use of the categorical duality between behaviors and finitely generated modules [39, 75] by which questions about the former could be faithfully carried over to questions about the latter. This translation which is a consequence of the cogenerator property of $\mathfrak{D}'$ and $\mathfrak{C}^\infty$ has to be modified even for the space of $\mathcal{S}'$ of temperate distributions, which although injective does not cogenerate.
In the spaces $\mathfrak{C}_c$, $\mathcal{E}'$ (compactly supported distributions) and $\mathcal{S}$ (the Schwartz space of rapidly decreasing functions), when the projection of a behavior may fail to be one [62], the question of implementation itself has to be reinterpreted. This chaper shows that a

natural weakening of the question does admit a solution, in fact the same solution as when the space is an injective cogenerator.
To overcome the problem arising from the loss of Oberst's duality, this chapter must rely on a PDE analogue of the Hilbert Nullstellensatz [60]. This statement is in terms of associated primes when the space is $\mathcal{S}'$, and is identical for the spaces $\mathfrak{C}_c, \mathcal{E}'$ and $\mathcal{S}$ (these six spaces $\mathfrak{D}', \mathfrak{C}^\infty, \mathcal{S}', \mathcal{S}, \mathcal{E}'$ and $\mathfrak{C}_c$, we collectively call the *classical spaces*). Thus for ease of exposition we write specifically for the space $\mathfrak{C}_c$. All results carry over, a fortiori, to the spaces $\mathcal{E}'$ and $\mathcal{S}$, while we confine ourselves to a few remarks about $\mathcal{S}'$.
The chapter is organized as follows – after recollecting the standard definition of regular implementation, we point out through examples the problems that now have to be overcome. This suggest a natural reformulation of the problem. The final choice of the controller equations (that will implement the controller) has to be chosen a little carefully – this requires the Nullstellensatz statement, here for $\mathfrak{C}_c$. The last section is devoted to the construction of the implementing controller.

4.2 Preliminaries

The problem of regular implementation considered in [57, 63] is solved there in two stages, first implementation and then the question of regularity. We follow the same pattern.
Let $\mathfrak{D} = \mathbb{C}[\partial_1, \ldots, \partial_n]$ be the ring of constant coefficient partial differential operators on $\mathbb{R}^n$. Let $\mathfrak{D}'$ be the space of distributions on $\mathbb{R}^n$ considered as an $\mathfrak{D}$-module, the module structure given by differentiation. Let $\mathcal{F}$ be an $\mathfrak{D}$-submodule of $\mathfrak{D}'$, here one of the six classical spaces already listed. Let $p(\partial) = (p_1(\partial), \ldots, p_k(\partial))$ be an element of $\mathfrak{D}^k$. The behavior $\mathfrak{B}_{\mathcal{F}}(p)$ of $p(\partial)$ in $\mathcal{F}$ is the kernel of the morphism

$$\begin{array}{rcl} p(\partial) : \mathcal{F}^k & \longrightarrow & \mathcal{F} \\ f = (f_1, \ldots, f_k) & \longmapsto & p(\partial) f = \sum p_i(\partial) f_i. \end{array}$$

Let P be a submodule of $\mathfrak{D}^k$. The behavior $\mathfrak{B}_{\mathcal{F}}(P)$ of P in $\mathcal{F}$ is $\cap_{p \in P} \mathfrak{B}_{\mathcal{F}}(p)$, the common behavior of all the $p \in P$. If $(p_1, \ldots, p_\ell)$, $p_i = (p_{i_1}(\partial), \ldots, p_{i_k}(\partial)$, is any set of generators for P, then writing the entries $p_{ij}(\partial)$ as a matrix $P(\partial)$, gives a morphism

$$\begin{array}{rcl} P(\partial) : \mathcal{F}^k & \longrightarrow & \mathcal{F}^\ell \\ f & \longmapsto & P(\partial) f \end{array}$$

whose kernel is the behavior $\mathfrak{B}_{\mathcal{F}}(P)$ (we omit the subscript $\mathcal{F}$ if it is clear from the context which space the behavior is located in).
Suppose there is given a natural splitting of the k coordinates of $\mathfrak{D}^k$, and thus of $\mathcal{F}^k$, into two sets of r and s elements, i.e. say

$$\mathfrak{D}^k \simeq \mathfrak{D}^r \oplus \mathfrak{D}^s$$

and the corresponding $\mathcal{F}^k = \mathcal{F}^r \oplus \mathcal{F}^s$. A submodule P_1 of $\mathfrak{D}^r$ is identified naturally with a submodule of $\mathfrak{D}^k$ via the inclusion $i_1 : \mathfrak{D}^r \longrightarrow \mathfrak{D}^r \oplus \mathfrak{D}^s$, $x \mapsto i_1(x) = (x, 0)$, and similarly for submodules $\mathfrak{D}^s$ via the inclusion $i_2 : \mathfrak{D}^s \longrightarrow \mathfrak{D}^r \oplus \mathfrak{D}^s$, $y \mapsto i_2(y) = (0, y)$. The submodule of $\mathfrak{D}^r \oplus \mathfrak{D}^s$ consisting of all $(x, 0)$ in P is denoted $\mathfrak{D}^r \cap P$ and is isomorphic to $i_1^{-1}(P)$. Similarly, $P \cap \mathfrak{D}^s$ is the submodule $\{(0, y) \in P\}$, isomorphic to $i_2^{-1}(P)$. The behavior $\mathfrak{B}(P_1)$ in $\mathcal{F}^r$ must now be identified with the behavior $\mathfrak{B}(P_1) \oplus \mathcal{F}^s$ in $\mathcal{F}^r \oplus \mathcal{F}^s$ (as the behavior of the 0 submodule of $\mathfrak{D}^s$ is $\mathcal{F}^s$), and similarly for the second factor.
Finally,let $\pi_1 and \pi_2$ denote the projections of $\mathcal{F}^r \sum F^s$ onto the first and second factors.

The problem of implementability: Let $\mathfrak{B}$ be a behavior in $\mathcal{F}^r \oplus \mathcal{F}^s$ (the full plant behavior) and $\mathcal{K}$ a behavior in $\mathcal{F}^r$ (the manifest behavior that must be attained). Is there a behavior $\mathfrak{C}$ in $\mathcal{F}^s$ (the controller behavior) such that

$$\pi_1(\mathfrak{B} \cap (\mathcal{F}^r \oplus \mathfrak{C})) = \mathcal{K} \quad ?$$

We refer to earlier work for the control theoretic significance of this problem, where it has been solved when $\mathcal{F}$ is $\mathfrak{D}'$ or $\mathfrak{C}^\infty$, that is, for injective cogenerators [63].

Suppose now that $\mathcal{F}$ is the space $\mathfrak{C}_c$ of compactly supported smooth functions. We then run into inmediate difficulties, even with the above formulation of the problem, as now a projection of a behavior need not to be a behavior [62].$\mathfrak{C}_c^2$

Example 4.2.1 : Let $\mathfrak{D} = \mathbb{C}[\frac{d}{dt}]$, and let $\pi_1 : \mathfrak{C}_c^2 \longrightarrow \mathfrak{C}_c$ be the projection onto the first factor.
Let P be the cyclic submodule of $\mathfrak{D}^2$ generated by $(1, -\frac{d}{dt})$. Then $\mathfrak{B}_{\mathfrak{D}}(P) = \{(\frac{df}{dt}, f) \mid f \in \mathfrak{C}_c\}$, but $\pi_1(\mathfrak{B}_{\mathfrak{D}}(P)) = \{\frac{df}{dt} \mid f \in \mathfrak{C}_c\}$ is not a behavior in $\mathfrak{C}_c$.

It is shown in [62] that even though a projection of a behavior need not to be one, the smallest behavior containing it can be characterized. More precisely, the following is true.

Proposition 4.2.2 *Let $\mathcal{F}$ be $\mathfrak{C}_c, \mathcal{E}'$ or $\mathcal{S}$. Let M be a submodule of $\mathfrak{D}^k$. Then $\mathfrak{B}_{\mathcal{F}}(i_1^{-1}(M)) \simeq \mathfrak{B}_{\mathcal{F}}(\mathfrak{D}^r \cap M)$ is the smallest behavior containing $\pi_1(\mathfrak{B}_{\mathcal{F}}(M))$. If $\mathcal{F}$ is an injective $\mathfrak{D}$-module (such as $\mathfrak{D}', \mathfrak{C}^\infty$ or $\mathcal{S}'$), then these two behaviors are equal so that the projection of a behavior is also a behavior.*

This suggest the following reformulation:

Implementability in a classical space: Let $\mathcal{F}$ be a classical space, and $\mathfrak{B}$ a behavior in $\mathcal{F}^r \oplus \mathcal{F}^s$. Let $\mathcal{K}$ be a (manifest) behavior in $\mathcal{F}^r$. Is there a controller behavior $\mathfrak{C}$ in $\mathcal{F}^s$ such that $\mathcal{K}$ is the smallest behavior containing $\pi_1(\mathfrak{B} \cap (\mathcal{F}^r \oplus \mathfrak{C}))$?

In the course of development of the results of this chapter, we have to sometimes consider the sum of two behaviors. For instance, to say that the controller $\mathfrak{C}$ implements *regularly* is to say that the controller equations C have nothing in common with the behavior equations M, i.e. $M \cap C = 0$. This is really a statement about behaviors, namely that $\mathfrak{B} + \mathfrak{C} = \mathcal{F}^k$. In general, the sum of two behaviors $\mathfrak{B}(P_1)$ and $\mathfrak{B}(P_2)$ need not to be a behavior, but always, the smallest behavior containing the sum is $\mathfrak{B}(P_1 \cap P_2)$ [61].

Example 4.2.3 : Let $\mathfrak{D} = \mathbb{C}[\frac{d}{dt}]$ and let P_1 and P_2 be cyclic submodules of $\mathfrak{D}^2$ generated by $(1, 0)$ and $(1, -\frac{d}{dt})$ respectively. Then $P_1 \cap P_2 = 0$, so that $\mathfrak{B}_{\mathfrak{C}_c}(P_1 \cap P_2) = \mathfrak{C}_c^2$. On the other hand, $\mathfrak{B}_{\mathfrak{D}}(P_1) = \{(0, f) \mid f \in \mathfrak{C}_c\}$ and $\mathfrak{B}_{\mathfrak{D}}(P_2) = \{(\frac{dg}{dt}, g) \mid g \in \mathfrak{C}_c\}$, and it is easy to see that $\mathfrak{B}_{\mathfrak{D}}(P_1) + \mathfrak{B}_{\mathfrak{D}}(P_2)$ is not all $\mathfrak{C}_c^2$. As always $\mathfrak{B}_{\mathfrak{D}}(P_1 \cap P_2)$ is the smallest behavior containing the sum, and thus it follows that here this sum is not a behavior.

A third difficulty encountered when the signal space is not an injective cogenerator is that there is no longer a bijective correspondence between behaviors in $\mathcal{F}^k$ and submodules of $\mathfrak{D}^k$ [60].

Example 4.2.4 : Let $\mathcal{F} = \mathfrak{C}_c$, and $P_1 = (\frac{d}{dt})$, $P_2 = (1)$ ideals of $\mathfrak{D}$. Then $\mathfrak{B}_{\mathfrak{D}}(P_1) = \mathfrak{B}_{\mathfrak{D}}(P_2) = 0$.

Thus, if P is a submodule of $\mathfrak{D}^k$ and $\mathfrak{B}_{\mathcal{F}}(P)$ its behavior in $\mathcal{F}^k$, then the submodule of all elements p in $\mathfrak{D}^k$ such that $\mathfrak{B}_{\mathcal{F}}(P) \subset \mathfrak{B}_{\mathcal{F}}(p)$ – which is in general larger than P – is called the (Willems) closure of P with respect to $\mathcal{F}$, and is denoted $\overline{P_{\mathcal{F}}}$. The calculation of this closure is analogous to the Hilbert Nullstellensatz, and is studied in [60] for the classical spaces. For the spaces $\mathfrak{C}_c, \mathcal{E}'$ and $\mathcal{S}$, this calculation is the following.

Proposition 4.2.5 *Let $\mathcal{F}$ be $\mathfrak{C}_c, \mathcal{E}'$ or $\mathcal{S}$, and let P be a submodule of $\mathfrak{D}^k$. Then the closure $\bar{P}$ of P with respect to these $\mathcal{F}$ equals $\{p \in \mathfrak{D}^k \mid ap \in P,\ a \neq 0\}$. In other words, if $\pi : \mathfrak{D}^k \longrightarrow \mathfrak{D}^k/P$ is the natural projection, and $\mathcal{T}$ is the torsion submodule of $\mathfrak{D}^k/P$, then $\bar{P} = \pi^{-1}(\mathcal{T})$.*

Remark 4.2.6 : The papers [60, 61] calculate the closure also with respect to the space $\mathcal{S}'$ of temperate distributions. As this calculation is somewhat technical (and involves the associate primes of P) we do not include it here. Thus apart from some remark about $\mathcal{S}'$, we confine ourselves to the spaces $\mathfrak{C}_c, \mathcal{E}'$ and $\mathcal{S}$.

Remark 4.2.7 : If P equals its closure $\bar{P}_{\mathcal{F}}$, then P is said to be *closed* with respect to $\mathcal{F}$. By definition the behavior of a submodule P equals that of its closure $\bar{P}_{\mathcal{F}}$; indeed $\bar{P}_{\mathcal{F}}$ is the largest submodule with the same behavior as that of P. Given a behavior $\mathfrak{B}$ we denote this largest submodule by $\mathcal{M}(\mathfrak{B})$ and call it the *the vanishing module* of $\mathfrak{B}$. Thus $\mathcal{M}(\mathfrak{B}_{\mathcal{F}}(P)) = \bar{P}_{\mathcal{F}}$.

4.3 Implementation

The first result of this section is a characterization of those behaviors $\mathcal{K}$ that are implementable when the underlying signal space is $\mathfrak{C}_c$ (but, as noted above, the result is equally valid for $\mathcal{E}'$ and $\mathcal{S}$).

Theorem 4.3.1 *Let $\mathfrak{B}$, the full plant behavior in $\mathfrak{C}_c^r \oplus \mathfrak{C}_c^s$, and $\mathcal{K}$ a behavior in $\mathfrak{C}_c^r$ be given. Let $M = \mathcal{M}(\mathfrak{B})$ be the vanishing module of the plant behavior. Then the following are equivalent:*

1. *There is a behavior $\mathfrak{C}$ in $\mathfrak{C}_c^s$ such that $\mathcal{K}$ is the smallest behavior containing $\pi_1(\mathfrak{B} \cap (\mathfrak{C}_c^r \oplus \mathfrak{C}))$ (so that if this projection is itself a behavior, then it equals $\mathcal{K}$).*

2. $\mathfrak{B}(\pi_1(M)) \subset \mathcal{K} \subset \mathfrak{B}(\mathfrak{D}^r \cap M)$.

3. *There is a submodule K of $\mathfrak{D}^r$ such that $\mathcal{K} = \mathfrak{B}(K)$ and $\mathfrak{D}^r \cap M \subset K \subset \pi_1(M)$. (In fact we can take $K = \mathcal{M}(\mathcal{K}) \cap \pi_1(M)$).*

Proof : (1)$\Rightarrow$(2). As $\mathfrak{C}$ is a behavior in $\mathfrak{C}_c^s$, it must satisfy $0 \subset \mathfrak{C} \subset \mathfrak{C}_c^s$. If $\mathfrak{C} = 0$, then $\mathfrak{B} \cap (\mathfrak{C}_c^r \oplus 0) = \{(w, 0) \in \mathfrak{B}\} \simeq i_1^{-1}(\mathfrak{B})$ where $i_1 : \mathfrak{C}_c^r \longrightarrow \mathfrak{C}_c^r \oplus \mathfrak{C}_c^s$ is the canonical inclusion. By Proposition 4.1 of [62] quoted earlier, $i_1^{-1}(\mathfrak{B}) = \mathfrak{B}(\pi_1(M))$.
Suppose now that $\mathfrak{C} = \mathfrak{C}_c^s$. Then $\mathfrak{B} \cap (\mathfrak{C}_c^r \oplus \mathfrak{C}_c^s) = \mathfrak{B}$. The projection $\pi_1(\mathfrak{B})$ may not be a behavior, but the smallest behavior containing it is $\mathfrak{B}(i_1^{-1}(M))$ (by Proposition 4.2 of [62] also quoted earlier). By the identification of Section 2, this is precisely $\mathfrak{B}(\mathfrak{D}^r \cap M)$.

(2)$\Rightarrow$ (3). Taking the vanishing module of the three behaviors in (2) gives

$$\mathcal{M}(\mathfrak{B}(\mathfrak{D}^r \cap M)) \subset \mathcal{M}(\mathcal{K}) \subset \mathcal{M}(\mathfrak{B}(\pi_1(M))).$$

As $M = \mathcal{M}(\mathfrak{B})$, M is closed (here with respect to $\mathfrak{C}_c$), and hence so is $\mathfrak{D}^r \cap M$. The above inclusions then become

$$\mathfrak{D}^r \cap M \subset \mathcal{M}(\mathcal{K}) \subset \overline{\pi_1(M)}.$$

Let $\mathcal{M}(\mathcal{K}) \cap \pi_1(M) = K$. Then $\mathfrak{B}(\mathcal{M}(\mathcal{K})) + \mathfrak{B}(\pi_1(M)) = \mathcal{K} + \mathfrak{B}(\pi_1(M)) \subset \mathfrak{B}(K)$ - in general this inclusion may be strict and $\mathfrak{B}(K)$ only the smallest behavior that contains this sum [61], but here, by (2), $\mathfrak{B}(\pi_1(M)) \subset \mathcal{K}$, and hence $\mathcal{K} + \mathfrak{B}(\pi_1(M))$ equals $\mathcal{K}$. Then, as $\mathfrak{B}(K)$ is the smallest behavior that contains $\mathcal{K}$, it must be equal to $\mathcal{K}$.

(3)$\Rightarrow$ (1). Given K satisfying (3), let $\tilde{M}(K) = \{(m_1, m_2) \in M \mid (m_1, 0) \in K\}$ (our convention requires us to consider submodules of $\mathfrak{D}^r$ as submodules of $\mathfrak{D}^r \oplus \mathfrak{D}^s$). Let $C_{can} = \pi_2(\tilde{M}(K))$ – this is the canonical controller associated to K, see [63]. Interconnecting the plant to the controller defined by C_{can} is to augment the laws of M by the laws of C_{can}; This yields the module $M + C_{can}$. The submodule $\mathfrak{D}^r \cap (M + C_{can})$ equals K, by the very construction of the canonical controller.
Let $\mathfrak{C} \subset \mathfrak{C}_c^s$ be the behavior of C_{can} considered as a submodule of $\mathfrak{D}^s$. Indentifying C_{can} with a submodule of $\mathfrak{D}^r \oplus \mathfrak{D}^s$ identifies $\mathfrak{C}$ with the behavior $\mathfrak{C}_c^r \oplus \mathfrak{C}$ in $\mathfrak{C}_c^r \oplus \mathfrak{C}_c^s$. The behavior of $M + C_{can}$ is then $\mathfrak{B}(M) \cap \mathfrak{B}(C_{can}) = \mathfrak{B} \cap (\mathfrak{C}_c^r \oplus \mathfrak{C})$. The projection $\pi_1(\mathfrak{B} \cap (\mathfrak{C}_c^r \oplus \mathfrak{C}))$ is in general not a behavior, but the smallest behavior containing it is the behavior of $\mathfrak{D}^r \cap (M + C_{can}) = K$ which equals $\mathcal{K}$. □

Remark 4.3.2 : Note that condition 2 of Theorem 4.3.1 differs from the usual characterization of implementability that states that the desired behavior $\mathcal{K}$ should be wedged in between the hidden behavior and the manifest plant behavior (see [74]). Although $\mathfrak{B}(\pi_1(M))$ is indeed the hidden behavior, $\mathfrak{B}(\mathfrak{D}^r \cap M)$ is only the smallest behavior that contains the projection $\pi_1(\mathfrak{B})$. The latter, in general, need not be a behavior.

Remark 4.3.3 : It is an easy check that the canonical controller $\pi_2(\tilde{M}(K))$ is equal to $(K+M)\cap\mathfrak{D}^s$ [63, 67].

Remark 4.3.4 : From the proof of the above theorem it is clear that in order to construct, even define, the canonical controller or any other implementing controller, it is necessary that a submodule defining the behavior $\mathcal{K}$ be contained in $\pi_1(M)$. A priori, this need not be so, and the above necessary condition must be forced as in the proof of the theorem. The next example illustrates this problem.

Example 4.3.5 : Let $\mathfrak{D} = \mathbb{C}[\frac{d}{dt}]$, $\mathfrak{B} = \mathfrak{B}_{\mathfrak{D}}(M) \subset \mathfrak{C}_c^3 \oplus \mathfrak{C}_c$ and $\mathcal{K} = \mathfrak{B}_{\mathfrak{D}}(K) \subset \mathfrak{C}_c^3$ where

$$M = \begin{pmatrix} 0 & \frac{d^2}{dt^2} - \frac{d}{dt} & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & -1 & 1 \end{pmatrix} \text{ and } K = \begin{pmatrix} 0 & \frac{d}{dt} - 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $\mathfrak{D}^3 \cap M \subset K \subset \overline{\pi_1(M)}$ but K is not contained in $\pi_1(M)$.

The above theorem solves the implementation problem by *partial* interconnection, where the term partial refers to the fact that the controller equations must be a submodule of $\mathfrak{D}^s \subset \mathfrak{D}^r \oplus \mathfrak{D}^s$. If the controller equations are not restricted in this way, i.e. if they could be an arbitrary submodule of $\mathfrak{D}^r \oplus \mathfrak{D}^s$, then the interconnection of such a controller is said to be *full*. Full interconnection problems are in general easier to solve. In Rocha [57], the problem of implementability by partial interconnection was converted to a *full* interconnection implementability problem. We now extend this result to the case when the signal space is $\mathfrak{C}_c$ (or E' or S). Again, as $\mathfrak{C}_c$ is not an injective cogenerator, its proof requires more care.

Theorem 4.3.6 *Let $\mathfrak{B}$ and $\mathcal{K}$ be given as in Theorem 4.3.1. Let $M = \mathcal{M}(\mathfrak{B})$ be the vanishing module of $\mathfrak{B}$. Then statement (1) of Theorem 4.3.1 is equivalent to:*
(4) There is a submodule K of $\mathfrak{D}^r$ such that (3) of theorem 4.3.1 holds and also such that $\mathfrak{B}(\pi_2(\tilde{M}(K))$ is implementable by full interconnection with respect to $\mathfrak{B}(M \cap \mathfrak{D}^s)$, where as before $\tilde{M}(K) = \{(m_1, m_2) \in M \mid (m_1, 0) \in K\}$.
Further, the implementation in (1) is regular if and only if it is so in (4).

To prove this theorem we first need:

Lemma 4.3.7 *Let $M \subset \mathfrak{D}^r \oplus \mathfrak{D}^s$ and $C \subset \mathfrak{D}^s$. Suppose that K is a submodule of $\mathfrak{D}^r$ such that $\mathfrak{D}^r \cap M \subset K \subset \pi_1(M)$. Then the following are equivalent:*
(a) $\mathfrak{D}^r \cap (M + C) = K$
(b) $(M + (C \cap \pi_2(M))) \cap \mathfrak{D}^s = \pi_2(\tilde{M}(K))$.
Further $M \cap C = 0$ if and only if the sum in (b) is a direct sum.

Proof : This is Lemma 13 in [63]. □

Proof of Theorem 4.3.6

(1)$\Rightarrow$ (4). Let $C = \mathcal{M}(\mathfrak{C})$ and let $K = \mathfrak{D}^r \cap (M + C)$. Then $\mathfrak{B}(K)$, being the smallest behavior containing $\pi_1(\mathfrak{B} \cap (\mathfrak{C}_c^r \oplus \mathfrak{C}))$, equals $\mathcal{K}$. Clearly $\mathfrak{D}^r \cap M \subset \mathfrak{D}^r \cap (M + C) \subset \pi_1(M)$, so that the assumption of Lemma 4.3.7 is satisfied. Then (2) holds. Let $C' = C \cap \pi_2(M)$. The behavior $\mathfrak{B}((M + C') \cap \mathfrak{D}^s) = \mathfrak{B}((M \cap \mathfrak{D}^s) + C') = \mathfrak{B}(M \cap \mathfrak{D}^s) \cap \mathfrak{B}(C')$ thus equals $\mathfrak{B}(\pi_2(\tilde{M}(K)))$, so that (4) holds.

For the second part of the proof, we need the following:

Lemma 4.3.8 *With the assumptions of Lemma 4.3.7 , and for $C \subset \pi_2(M)$, $\overline{(M + C)} \cap \mathfrak{D}^s = \overline{\pi_2(\tilde{M}(K))}$ implies $\mathfrak{D}^r \cap \overline{(M + C)} = \bar{K}$, where the overline denotes closure with respect to $\mathfrak{C}_c$.*

Proof : Let x be in $\bar{K}$ so that for some nonzero a in $\mathfrak{D}$, $ax = (m_1, 0)$ is in K. Then, as K is contained in $\pi_1(M)$, there is an m_2 such that (m_1, m_2) is in $\tilde{M}(K)$, so that $(0, m_2)$ is in $\pi_2(\tilde{M}(K))$. Now the assumption $\overline{\pi_2(\tilde{M}(K))} = \overline{(M + C)} \cap \mathfrak{D}^s$ implies that there is a nonzero b in $\mathfrak{D}$ such that $(0, bm_2)$ equals $(0, m_2') + (0, c)$, where $(0, c)$ is in C and $(0, m_2')$ in M.
Consider now $abx = (bm_1, 0)$ in K. By the above, $(bm_1, 0) = (bm_1, bm_2) - (0, bm_2) = (bm_1, bm_2) - (0, c) - (0, m_2')$ is in $M + C$, and hence in $\mathfrak{D}^s \cap (M + C)$. As $\mathfrak{D}$ is an integral domain, $ab \neq 0$. Hence x is in $\overline{\mathfrak{D}^s \cap (M + C)} = \mathfrak{D}^s \cap \overline{(M + C)}$ (the closure of a finite intersection is the intersection of the closures [61]).
Conversely, suppose that x is in $\mathfrak{D}^r \cap \overline{(M + C)}$, then for some nonzero a, ax is in $\mathfrak{D}^r \cap (M + C)$. Thus ax is of the form $(m, c) + (0, -c)$, where (m, c) is in M and $(0, -c)$ is in C. This $(0, -c)$ is also in $(M + C) \cap \mathfrak{D}^s$, which implies that $(0, -c)$ is in $\overline{\pi_2(\tilde{M}(K))}$ by the assumption of the lemma.
Again, by definition of the closure with respect to $\mathfrak{C}_c$, there is a nonzero b in $\mathfrak{D}$ such that $(0, -bc)$ is in $\pi_2(\tilde{M}(K))$. By Remark 4.3.2 following Theorem 4.3.1, $\pi_2(\tilde{M}(K))$ equals $(K + M) \cap \mathfrak{D}^s$. Thus it follows that $(0, -bc) = (-m_1, 0) + (m_1, -bc)$ with $(-m_1, 0)$ in K and $(m_1, -bc)$ in M. This then implies that $abx = (bm, bc) + (0, -bc) = (bm, bc) + (m_1, -bc) + (-m_1, 0) = (bm + m_1, 0) + (-m_1, 0)$ where $(bm + m_1, 0)$ is in $\mathfrak{D}^r \cap M$ and hence by assumption (of Lemma 4.3.7) in K. Thus abx is in K, and again as $ab \neq 0$, this implies that x is in $\bar{K}$. □

Proof of Theorem 4.3.6 continued : (4)$\Rightarrow$(1). Let $\mathfrak{C}$ be the behavior which implements $\mathfrak{B}(\pi_2(\tilde{M}(K)))$ by full interconnection with respect to $\mathfrak{B}(M \cap \mathfrak{D}^s)$, that is $\mathfrak{B}(M \cap \mathfrak{D}^s) \cap \mathfrak{C} = \mathfrak{B}(\pi_2(\tilde{M}(K))$. Let $\mathcal{M}(\mathfrak{C}) = C$, then it follows that $\mathfrak{B}(M \cap \mathfrak{D}^s) + C) = \mathfrak{B}(M \cap \mathfrak{D}^s) \cap \mathfrak{C} = \mathfrak{B}(\pi_2(\tilde{M}(K))$, so that taking the vanishing module of both these behaviors gives

$$\overline{(M \cap \mathfrak{D}^s) + C} = \overline{(M + C)} \cap \mathfrak{D}^s = \overline{\pi_2(\tilde{M}(K))}$$

(where the first equality is the "modular law"). By Lemma 4.3.8, it follows that $\mathfrak{D}^r \cap \overline{(M + C)} = \bar{K}$. The behavior of $\mathfrak{D}^r \cap \overline{(M + C)}$ equals that of $\mathfrak{D}^r \cap (M + C)$ – by definition

of closure – and this is the smallest behavior containing $\pi_1(\mathfrak{B} \cap (\mathfrak{C}_c^r \oplus \mathfrak{C}))$ [62]. This establishes (1).
To prove regularity we need the elementary.

Lemma 4.3.9 *Let F be a classical space. Let M, M' and C, C' be submodules of $\mathfrak{D}^k$ such that $\mathfrak{B}_{\mathcal{F}}(M) = \mathfrak{B}_{\mathcal{F}}(M')$ and $\mathfrak{B}_{\mathcal{F}}(C) = \mathfrak{B}_{\mathcal{F}}(C')$. Then $M \cap C = 0$ if and only if $M' \cap C' = 0$*

Proof : For any classical space F, the only submodule of $\mathfrak{D}^k$ whose behavior is all of $\mathcal{F}^k$ is the 0 submodule. Thus if $M \cap C = 0$, then $\mathcal{F}^k$ is the smallest behavior containing $\mathfrak{B}_{\mathcal{F}}(M) + \mathfrak{B}_{\mathcal{F}}(C)$, and hence also the smallest behavior containing $\mathfrak{B}_{\mathcal{F}}(M') + \mathfrak{B}_{\mathcal{F}}(C')$. This implies that $M' \cap C' = 0$. □

Conclusion of the proof of theorem 4.3.6: It only remains to prove regularity, and this now follows from Lemma 4.3.7. By Lemma 4.3.9, the choice of the submodules M and C which give the behaviors $\mathfrak{B}, \mathfrak{C}$ is irrelevant. □